Calculus 1
Final Exam – Solutions + Marking Scheme
October 27, 2026 (18:15 – 20:15)

university of groningen

1) Solve the initial value problem $y'(x) + 3x^2\, y(x) = x^2$, $y(0) = 1$.

Solution. Method #1 (Separation of variables). By rearranging the equation as $y' = (1 - 3y)x^2$, we notice that it is a separable ODE. Dividing both sides by $3y - 1$ yields

$$\frac{y'}{3y - 1} = -x^2.$$

Note that the division by $3y - 1$ implicitly ignores the constant function $y(x) = 1/3$, which is a valid solution of the ODE $y' + 3x^2\, y = x^2$. However, this does not cause a problem as the initial value $y(0) = 1 \neq 1/3$ discards this solution. By integrating both sides with respect to x, we get

$$\int \frac{y'}{3y - 1}\, dx = \int (-x^2)\, dx$$

which evaluates to

$$\frac{1}{3} \ln |3y - 1| = -\frac{x^3}{3} + C \quad \Rightarrow \quad y(x) = \frac{1}{3} \pm \frac{e^{3C}}{3}\, e^{-x^3}.$$

Therefore the general solution reads

$$y(x) = \frac{1}{3} + Ae^{-x^3},$$

where A is an arbitrary constant. To find the particular solution, we employ the initial value

$$1 = y(0) = \frac{1}{3} + Ae^{-0^3} = \frac{1}{3} + A(1) = \frac{1}{3} + A \quad \Rightarrow \quad A = \frac{2}{3}.$$

In summary, the solution of the initial value problem $y'(x) + 3x^2\, y(x) = x^2$, $y(0) = 1$ is

$$y(x) = \frac{1 + 2e^{-x^3}}{3}.$$

Method #2 (Integrating factor). The ODE, being of the form $y' + P(x)y = Q(x)$, is a linear first-order equation, therefore we may use the integrating factor method. The integrating factor reads

$$I(x) = e^{\int P(x)\, dx} = e^{\int 3x^2\, dx} = e^{x^3}.$$

Multiplying both sides of the ODE by I turns the equation into

$$y'e^{x^3} + y(3x^2e^{x^3}) = x^2e^{x^3}.$$

Note that the left-hand side is the derivative of ye^{x^3} (by the Product Rule), i.e.

$$(ye^{x^3})' = x^2e^{x^3}.$$

Integration on both sides with respect to x leads to the following equation

$$ye^{x^3} = \int x^2e^{x^3}\, dx.$$

By performing the integral on the right-hand side (e.g. via substituting $u = x^3$), we find that

$$ye^{x^3} = \frac{1}{3}e^{x^3} + A$$

and therefore the general solution is found

$$y(x) = \frac{1}{3} + Ae^{-x^3},$$

where A is an arbitrary constant. To find the particular solution, we employ the initial value

$$1 = y(0) = \frac{1}{3} + Ae^{-0^3} = \frac{1}{3} + A(1) = \frac{1}{3} + A \quad \Rightarrow \quad A = \frac{2}{3}.$$

In summary, the solution of the initial value problem $y'(x) + 3x^2\, y(x) = x^2$, $y(0) = 1$ is

$$y(x) = \frac{1 + 2e^{-x^3}}{3}.$$

Method #3 (Variation of constant). The ODE, being of the form $y' + P(x)y = Q(x)$, is a linear first-order equation, therefore we may use the variation of constant method. We first solve the homogeneous equation (which is a separable ODE)

$$y_h' + 3x^2 y_h = 0,$$

We find the general solution of the homogeneous equation to be

$$y_h(x) = e^{\int (-3x^2)\, dx} = e^{-x^3 + C} = Ae^{-x^3}.$$

The solution of the non-homogeneous equation is obtained by varying the parameter A, that is looking for y in the form

$$y(x) = A(x)e^{-x^3}.$$

Plugging this back into the original ODE, we find that

$$A'(x)e^{-x^3} = x^2 \quad \Rightarrow \quad A'(x) = x^2 e^{x^3} \quad \Rightarrow \quad A(x) = \int x^2 e^{x^3}\, dx = \frac{1}{3}e^{x^3} + \tilde{C}.$$

Therefore the general solution reads

$$y(x) = \frac{1}{3} + \tilde{C}e^{-x^3},$$

where $\tilde{C}$ is an arbitrary constant. To find the particular solution, we employ the initial value

$$1 = y(0) = \frac{1}{3} + \tilde{C}e^{-0^3} = \frac{1}{3} + \tilde{C}(1) = \frac{1}{3} + \tilde{C} \quad \Rightarrow \quad \tilde{C} = \frac{2}{3}.$$

In summary, the solution of the initial value problem $y'(x) + 3x^2\, y(x) = x^2$, $y(0) = 1$ is

$$y(x) = \frac{1 + 2e^{-x^3}}{3}.$$

2) Find the area of the region that lies under the curve $x^2 y = e^{A/x}$ (with $A < 0$) in the first quadrant of the xy-plane.

Solution. Solving the equation $x^2 y = e^{A/x}$ for y allows us to view the region in question as that below the graph of the function $f(x) = x^{-2}e^{A/x}$, $x > 0$. Consequently, its area can be calculated via the improper integral of $f(x)$ over the interval $(0, \infty)$, that is

$$I = \int_0^\infty f(x)\, dx = \int_0^\infty x^{-2} e^{A/x}\, dx.$$

(Note that $f(x) > 0$ for all $x > 0$.) The appearance of negative powers of x in the integrand impels us to substitute $u = A/x$ ($du = -Ax^{-2}\, dx$) and due to $A < 0$ obtain the u-integral over the interval $(-\infty, 0)$ resulting in

$$I = -\frac{1}{A} \int_{-\infty}^0 e^u\, du.$$

By definition, this means that the following limit at infinity is taken

$$I = -\frac{1}{A}\lim_{a\to-\infty}\int_a^0 e^u\,du = -\frac{1}{A}\lim_{a\to-\infty}[e^u]_a^0 = -\frac{1}{A}\lim_{a\to-\infty}(1-e^a) = -\frac{1}{A} = \frac{1}{|A|}.$$

3) Use Taylor Series to find the limit $\lim\limits_{x\to0}\dfrac{4\sin(2x)-2\sin(4x)}{3x-\arctan(3x)}$.

Solution. Let us begin by recalling the Taylor series (centred at 0) of the sine and inverse tangent functions. We have

$$\sin(\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n+1}}{(2n+1)!} = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \ldots, \qquad \forall\ \alpha\in\mathbb{R}$$

and

$$\arctan(\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n+1}}{(2n+1)} = \alpha - \frac{\alpha^3}{3} + \frac{\alpha^5}{5} - \ldots \qquad |\alpha|\le 1.$$

Substituting $\alpha = 2x$ and $\alpha = 4x$ in the sine series and $\alpha = 3x$ in the inverse tangent series, we obtain

$$\frac{4\sin(2x)-2\sin(4x)}{3x-\arctan(3x)} = \frac{4(2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\ldots)-2(4x-\frac{64x^3}{3!}+\frac{1024x^5}{5!}-\ldots)}{3x-(3x-\frac{27x^3}{3}+\frac{243x^5}{5}-\ldots)}$$

Assuming that we are within the interval of convergence, we may manipulate (add, multiply, simplify) series as we do for sums with finitely many terms. Therefore we get

$$\frac{4\sin(2x)-2\sin(4x)}{3x-\arctan(3x)} = \frac{16x^3-16x^5+\ldots}{9x^3-\frac{243x^5}{5}+\ldots} = \frac{16-16x^2+\ldots}{9-\frac{243x^2}{5}+\ldots}$$

Finally, we may let x tend to zero and use continuity to evaluate the limit by direct substitution

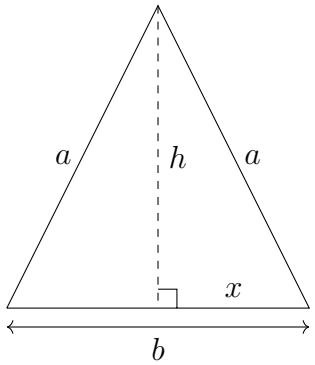
$$\lim_{x\to0}\frac{4\sin(2x)-2\sin(4x)}{3x-\arctan(3x)} = \frac{16-16(0^2)+\ldots}{9-\frac{243(0^2)}{5}+\ldots} = \frac{16}{9},$$

where all but the constant terms remained.

4) Among all isosceles triangles of a fixed perimeter, show that the equilateral triangle has the greatest area.

Solution. Let us introduce some notation (cf. the figure):

- a (length of sides),
- $b = 2x$ (length of base),
- h (length of altitude to base).



The total perimeter of the triangle is of length $L = 2a + b = 2(a + x)$. The area of the triangle is

$$A = \frac{bh}{2} = xh = x\sqrt{a^2-x^2},$$

where we use the Pythagorean theorem. The relation $a = \frac{L}{2} - x$ lets us express the area as a function of x alone (n.b. the length L is fixed/constant):

$$A(x) = x\sqrt{\left(\tfrac{L}{2}-x\right)^2-x^2} = x\sqrt{\tfrac{L^2}{4}-Lx}.$$

Note that by the triangle inequality ($b \le a + a = 2a$) we must have $0 \le x \le \frac{L}{4}$. Of course, the area vanishes at the end-point values $x = 0$, $x = \frac{L}{4}$ (as the triangle becomes a line segment there). Also note that $A(x)$ is differentiable for $x \in (0, \frac{L}{4})$. Therefore if the area is maximal, we must have $A'(x) = 0$.

$$A'(x) = \sqrt{\tfrac{L^2}{4} - Lx} - \frac{Lx}{2\sqrt{\frac{L^2}{4} - Lx}} = \frac{2(\frac{L^2}{4} - Lx) - Lx}{2\sqrt{\frac{L^2}{4} - Lx}} = 0 \quad \Leftrightarrow \quad \tfrac{L^2}{2} - 3Lx = 0 \quad \Leftrightarrow \quad x = \tfrac{L}{6}.$$

The above calculation also shows that the first derivative $A'(x)$ changes sign from positive to negative as x is increased across $\frac{L}{6}$. By the First Derivative Test, $A(x)$ has a (local) maximum at $x = \frac{L}{6}$. By the Closed Interval Method applied over $x \in [0, \frac{L}{4}]$, this maximum is found to be absolute. Finally, note that for $x = \frac{L}{6}$, we have

$$a = \tfrac{L}{2} - \tfrac{L}{6} = \tfrac{L}{3}, \quad b = 2(\tfrac{L}{6}) = \tfrac{L}{3}$$

that is all three sides of the triangle have equal length. We showed that the greatest area is realized by the equilateral triangle.

5) Solve the following initial value problem

$$y''(x) + 3y'(x) + 2y(x) = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution. The auxiliary equation for the above ODE can be written as

$$r^2 + 3r + 2 = 0.$$

This quadratic has distinct real roots, namely

$$r_1 = -1, \quad r_2 = -2.$$

Therefore the general solution of the differential equation takes the following form

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

To find the values of the constants c_1, c_2 we need to compute the derivative of $y(x)$:

$$y'(x) = -c_1 e^{-x} - 2c_2^{-2x}$$

Therefore the initial values yield the following system of linear equations for c_1, c_2:

$$1 = y(0) = c_1 + c_2$$

$$2 = y'(0) = -c_1 - 2c_2$$

The unique solution is $c_1 = 4, c_2 = -3$. In conclusion, the solution of the initial value problem $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 2$ is

$$y(x) = 4e^{-x} - 3e^{-2x}.$$

6) Calculate the volume of the solid obtained by rotating the curve $y = \tan(x/2), \quad 0 \le x \le \pi/2$ about the x-axis.

Solution. The volume in question is given by the definite integral

$$V = \pi \int_0^{\pi/2} y^2 \, dx = \pi \int_0^{\pi/2} \tan^2(x/2) \, dx.$$

Method #1 (Trigonometric identities). The trigonometric identity $\tan^2(\alpha) = \sec^2(\alpha) - 1$ lets us write

$$V = \pi \int_0^{\pi/2} \left(\sec^2(x/2) - 1 \right) dx.$$

The trigonometric derivative $[\tan(x)]' = \sec^2(x)$ combined with the Chain Rule and the Fundamental Theorem of Calculus leads to the result:

$$V = \pi\big[2\tan(x/2) - x\big]_0^{\pi/2} = \pi\big(2\tan(\pi/4) - \pi/2\big) = 2\pi - \pi^2/2,$$

where we used the fact that $\tan(\pi/4) = 1$.

Method #2 (Weierstrass substitution). The tangent half-angle substitution $u = \tan(x/2)$ transforms the integral (via $dx = 2du/(1+u^2)$) into an algebraic u-integral over the interval $(0, 1)$:

$$V = \pi\int_0^1 \frac{2u^2}{1+u^2}\,du = 2\pi\int_0^1 \frac{u^2}{1+u^2}\,du = 2\pi\int_0^1 \frac{1+u^2-1}{1+u^2}\,du = 2\pi\int_0^1 \left(1 - \frac{1}{1+u^2}\right)du.$$

The inverse trigonometric derivative $[\arctan(u)]' = 1/(1+u^2)$ combined with the Chain Rule and the Fundamental Theorem of Calculus leads to the result:

$$V = 2\pi\big[u - \arctan(u)\big]_0^1 = 2\pi\big(1 - \arctan(1)\big) = 2\pi - \pi^2/2,$$

where we used the fact that $\arctan(1) = \pi/4$.